The 1st International and 3rd National Conference on Biomathematics
19-21 January 2022, Damghan University, Iran.

Application of Lie symmetries to construct new conservation laws of Hunter-Saxton equation

Mehdi Jafari, Amirhesam Zaeim, Somayeh alsadat Mahdiun

[1] Corresponding author. Department of Mathematics, Payame Noor University, PO BOX 19395-3697, Tehran, Iran. E-mail: m.jafarii@pnu.ac.ir
[2] Department of Mathematics, Payame Noor University, PO BOX 19395-3697, Tehran, Iran. E-mail: zaeim@pnu.ac.ir
[3] Department of Mathematics, Payame Noor University, PO BOX 19395-3697, Tehran, Iran. E-mail: ssm.mahdion@gmail.com

***Abstract*— In this paper, determining equation for multipliers and the 2-dimensional homotopy formula employed to construct higher order conservation laws for the Hunter-Saxton equation(HSE). Furthermore,The invariance properties of the multipliers with respect to the Lie point symmetry generators are investigated to construct new conservation laws.**

Keywords—Lie Symmetries, Conversation laws, Multiplier method, Homotopy operator, Hunter-Saxton equation(HSE).

I. Introduction

In the study of PDEs, conservation laws are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions[2, 3, 7, 8]. conversation laws are essential in physics, which state that specific quantities of an isolated system will remain constant over time.
Many methods for dealing with the conservation laws are derived, such as the method based on the Noethers theorem, the multiplier method, the Herman-Poole method, etc.[2, 3, 6, 7]. There are several limitations inherent in using Noethers theorem to find local conservation laws for a given PDE system. First of all, it is restricted to variational systems. Consequently, the linearizing operator (Frechet derivative) for PDE system, must be self-adjoint, which implies that the number of PDEs must be the same as the number of dependent variables appearing in system. In addition, one must find an explicit Lagrangian $L[U]$ whose Euler-Lagrange equations yield PDE system. There is also the difficulty of finding the variational symmetries of a given variational PDE system. The Herman-Poole method has some limitations such

as finding densities because densities are linear combinations of scaling symmetries with undetermined coefficients. So with out scaling symmetries we can not find the conservation laws. In[5] we improved multiplier method by using homotopy operator [8] as a powerful algorithmic tool, to calculate the conserved quantities (fluxes). Our aim is to continue, by analyzing the symmetry action of the Lie point symmetry generators on the multipliers of the Hunter- Saxton equation (HSE) to construct new conservation laws from known conservation laws.

$$(u_t + uu_x)_x = \frac{1}{2}u_x^2, \tag{1}$$

The HSE is a well known nonlinear hyperbolic PDE in mathematical physics. This equation has been first suggested by Hunter and Saxton[10] for the theoretical modeling of nematic liquid crystals. Liquid crystals are intermediate states of matter observed between the liquid and solid states. They have a fundamental role in indicating the characteristics of fluid flow. Usually, two linearly independent vector fields are needed for the complete description of nematic liquid crystals; One for characterizing the fluid flow and one for describing the orientation of the molecules, which is the so-called director field [9]. Also, the skeleton of the present paper is as follows. In section 2,We have referred to some definitions and previous results that are used in the later sections. Section 3 deals with the application of Lie symmetries to generate new multipliers of the conversation laws. we obtain new conservation laws of this equation by using the Lie symmetries in section 4.

II. Conservation Laws

Consider a nonlinear system $\Delta(x, u^{(n)}) = 0$ of partial differential equations of order n with p independent variables $x = (x^1, \dots, x^p)$ and q dependent variables $u = (u^1, \dots, u^q)$. A Conservation law of a PDF system is a divergence expression

$$\mathrm{Div}P = D_1P_1 + \cdots + D_pP_p = 0 \tag{2}$$

holding for all solutions $u = f(x, t)$of the given system. In (2),$P_i(x, u^{(r)}), i = 1, \cdots, p$, are called the fluxes of the conservation law, and the highestorder derivative r present in the fluxes is called the order of the conservation law. If one of the independent variables of PDE system is time t, the conservation law (2) takes the from

$$D_t\mathrm{T} + \mathrm{DivX} = 0, \tag{3}$$

where Div is the spatial divergence of X with respect to the spatial variables $x = (x^1, \dots, x^p)$. Here T referred to as a density, and $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ as spatial fluxes of the conservation law (2). The conserved density, T, and the associated flux, $\mathrm{X} = (\mathrm{X}_1, \cdots, \mathrm{X}_p)$ are functions of x, t, u and the derivatives of u with respect to both x and t. In particular, every admitted conservation law arises from multipliers $\xi^\nu(x, u^{(l)})$ such that

$$\xi^\nu(x, u^{(l)}) \cdot \Delta(x, u^{(n)}) = D_iP_i(x, u^{(r)}) \tag{4}$$

holds identically, where the summation convention is used whenever appropriate. Through this approach, the determining of conservation laws for a given PDE system reduces to finding sets of multipliers. The Euler operator with respect to u^α is the operator defined by

$$\mathbf{E}_{u^\alpha(x)}f = \sum_{k=0}^{M_1^\alpha}(-D_x)^k \frac{\partial f}{\partial u_{kx}^\alpha} \ \alpha = 1, ..., q, \tag{5}$$

where M_1^α is the order of f in u^α with respect to x. It is well known that, the Euler operators (5) annihilate any divergence expression $D_iP_i(x,u^{(r)})$. Thus, the following identities hold for arbitrary function u:

$$\mathbf{E}_{u^\alpha(x)}\big(D_iP_i(x,u^{(r)})\big)=0,\ \alpha=1,\cdots,q$$

The converse also holds. Specifically, the only scalar expressions annihilated by Euler operators are divergence expressions. In continuation, the following theorem is applied which connecting multipliers and conservation laws. A set of multipliers $\{\xi^\nu(x,u^{(l)})\}_{\nu=1}^N$ yields a conservation law for the PDE system if and only if the set of identities

$$\mathbf{E}_{u^\alpha(x)}\big(\xi^\nu(x,u^{(l)})\cdot\Delta(x,u^{(n)})\big)=0,\ \alpha=1,\cdots,q \tag{6}$$

holds identically. The set of equations (6) yields the set of linear determining equations to find all sets of conservation law multipliers of the PDE system by considering multipliers of all orders. To calculate the conserved quantities T and X, we need to invert the total divergence operator. This requires the integration (by parts) of an expression in multidimensions involving arbitrary functions and its derivatives, which is a difficult and cumbersome task. The homotopy operator is a powerful algorithmic tool (explicit formula) that originates from homological algebra and variational bicomplexes. $(\mathcal{H}^{(x)}_{u(x,t)}f,\mathcal{H}^{(t)}_{u(x,t)}f)$ that are defined as below, are the components of a 2- dimensional homotopy operator

$$\mathcal{H}^{(m)}_{u(x,t)}f=\int_0^1(\sum_{\alpha=1}^{q}\mathcal{I}^{(m)}_{u^\alpha(x,t)}f)[\lambda u]\frac{d\lambda}{\lambda},\ where\ m=x,t,. \tag{7}$$

The x-integrand $\mathcal{I}^{(x)}_{u^\alpha(x,t)}f$ and t- integrand, $\mathcal{I}^{(t)}_{u^\alpha(x,t)}f$ are defined as

$$\begin{aligned}\mathcal{I}^{(x)}_{u^\alpha(x,t)}f&=\sum_{k_1=1}^{M_1^\alpha}\sum_{k_2=0}^{M_2^\alpha}\Big(\sum_{i_1=0}^{k_1-1}\sum_{i_2=0}^{k_2}B^{(x)}u^\alpha_{i_1xi_2t}(-D_x)^{k_1-i_1-1}(-D_t)^{k_2-i_2}\Big)\frac{\partial f}{\partial u^\alpha_{k_1xk_2t}},\\ \mathcal{I}^{(t)}_{u^\alpha(x,t)}f&=\sum_{k_1=0}^{M_1^\alpha}\sum_{k_2=1}^{M_2^\alpha}\Big(\sum_{i_1=0}^{k_1}\sum_{i_2=0}^{k_2-1}B^{(t)}u^\alpha_{i_1xi_2t}(-D_x)^{k_1-i_1}(-D_t)^{k_2-i_2-1}\Big)\frac{\partial f}{\partial u^\alpha_{k_1xk_2t}},\end{aligned} \tag{8}$$

where M_1^α, M_2^α are the order of f in u^α with respect to x and t respectively, and combinatorial coefficient $B^{(x)}=B(i_1,i_2,k_1,k_2)$ defined as

$$B^{(x)}=B(i_1,i_2,k_1,k_2)=\frac{\binom{i_1+i_2}{i_1}\binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+k_2}{k_1}}, \tag{9}$$

In a similar way,$B^{(t)}=B(i_2,i_1,k_2,k_1)$is based on cyclic permutations. Suppose that an exact differential function $f=f(x,u^{(M)}(x))$ is given. That is, there exists a function $\mathbf{J}=\mathbf{J}(x,u^{(M-1)}(x))$ such that $f=\mathrm{Div}\mathbf{J}$. Then,

$$\mathbf{J}=\mathrm{Div}^{-1}f=(\mathcal{H}^{(x)}_{u(x,t)}f,\mathcal{H}^{(t)}_{u(x,t)}f).$$

III. Application of Lie symmetries to generate new multiplier

If a given PDE system is mapped into another PDE system by an invertible transformation (point or contact transformation) then each conservation law of first system, is transformed to a corresponding conservation law of second syatem. When the invertible transformation is a symmetry (discrete or continuous) of PDE system, then the corresponding conservation law is a conservation law of itself. Related to this, two formulas are presented by Bluman, Temuerchaolu and Anco in [2]. In this section, we use same idea for multipliers instead of conservation laws. By calculating the Lie point symmetry generators, we observed that in some cases the symmetry properties of the multipliers gave rise to alternative multipliers. This implies that alternative conservation laws can be calculated. Suppose $D_iP_i[U] = 0$ is a coservation law of PDE system. Under point transformation there exist functions $\{\Psi_i[W]\}_{i=1}^n$ such that formula

$$J[W]D_iP_i[U] = D_i\Psi_i[W] \tag{10}$$

holds, where $\Psi_i[W]$ is given explicitly in terms of the determinant obtained by replacing the ith column of the Jacobian determinant

$$J[W] = \frac{D(x^1, \cdots, x^n)}{D(z^1, \cdots, z^n)} \tag{11}$$

by column$[P_1[U], \cdots, P_n[U]]$.[3]
We now restrict our attention to the most important situation when the invertible point transformation is a symmetry of PDE system. If the invertible point transformation $(x; u) \longrightarrow (\tilde{x}(x; u); \tilde{u}(x; u))$ is a symmetry of the PDE system, then a conservation law $D_iPi[U] = 0$ of (1) yields the conservation law $D_i\Psi_i[U] = 0$ of PDE system. A set of multipliers $\{\bar{\lambda}(x, U, \partial_U, \cdots, \partial_{U^r})\}_{\nu=1}^n$ yields a new conservation law of PDE system $\Delta_\nu(x; u^{(k)})$,if and only if this set is independent of $\{\lambda_\nu(x, U, \partial_U, \cdots, \partial_{U^r})\}_{\nu=1}^n$ on all solutions $U(x) = u(x)$ of PDE system $\Delta_\nu(x; u^{(k)})$.

Proof. Two conservation laws of a PDE system $\Delta_\nu(x; u^{(k)})$ are equivalent if and only if their corresponding fluxes differ by a curl term on all solutions $U(x) = u(x)$ of PDE system $\Delta_\nu(x; u^{(k)})$. For a PDE system $\Delta_\nu(x; u^{(k)})$ in Cauchy-Kovalevskaya form, all equivalent conservation laws have the same set of multipliers when the multipliers are restricted to solutions $U(x) = u(x)$of $\Delta_\nu(x; u^{(k)})$ [1]. Hence two sets of multipliers are equivalent when they agree on all solutions $U(x) = u(x)$of $\Delta_\nu(x; u^{(k)})$. In particular, there is a one-to-one correspondence between nontrivial conservation laws (up to equivalence) and sets of nontrivial multipliers.

IV. New Conservation Laws for the HSE

In this section, we construct higher order conservation laws for the HSE. Consider the multipliers of the form $\xi(x, t, u, u_x, u_t, u_{xx}, u_{tt})$ for the (1). The determining equation for multipliers is

$$E_{u^\alpha}[\xi(x, t, u, u_x, u_t, u_{xx}, u_{tt}) \cdot (u_{xt} + \frac{1}{2}u_x^2 + uu_{xx})] \equiv 0 \tag{12}$$

Therefore, after straightforward but tedious calculation, we conclude that

$$\xi = \frac{1}{2}\frac{1}{(e^{-\frac{c_1}{u_x}})^2}\Big(2c_5c_6(-c_1 + u_x)e^{-c_1t} + 2x(c_1t + c_2)u_x + (c_1t^2 + 2(c_2 + 2c_3)t \\ +2c_4)u_t + 2c_3u - 2xc_1)(e^{-\frac{c_1}{u_x}}) + 2e^{-c_1t}c_5c_7(c_1 + u_x)\Big) \tag{13}$$

where $c_1, \cdots, c_7$ are constants. Conservation laws of the HSE obtained as follows

case1: $\xi_1 = \frac{1}{2}t^2u_t + xtu_x - x$

Therefore, we obtain the following conserved vector

$$\begin{aligned}
\mathrm{T}_1 &= \frac{1}{4}t^2uu_{xt} + \frac{1}{12}t^2u_x^2u + \frac{1}{6}t^2u^2u_{2x} + \frac{1}{8}t^2u_tu_x + \frac{1}{4}xtu_x^2 - \frac{1}{2}xu_x - \frac{1}{8}t^2uu_{tx} \\
&- \frac{1}{4}tuu_x - \frac{1}{4}xtuu_{2x} + \frac{1}{2}u, \\
\mathrm{T}_2 &= \frac{1}{3}t^2uu_xu_t + \frac{1}{2}xtuu_{xt} + \frac{1}{2}xtuu_x^2 - \frac{5}{4}xuu_x + \frac{1}{8}t^2u_t^2 + \frac{1}{4}xtu_xu_t - \frac{1}{2}xu_t - \frac{1}{4}tuu_t \\
&- \frac{1}{8}t^2uu_{2t} - \frac{1}{3}tu^2u_x - \frac{1}{6}t^2uu_tu_x + \frac{1}{2}xuu_x + \frac{1}{2}u^2.
\end{aligned}$$

case2: $\xi_2 = tu_t + xu_x$

Therefore, the corresponding conserved vector is:

$$\begin{aligned}
\mathrm{T}_1 &= \frac{1}{2}tuu_{xt} + \frac{1}{6}tuu_x^2 + \frac{1}{3}tu^2u_{2x} + \frac{1}{4}tu_tu_x + \frac{1}{4}xu_x^2 - \frac{1}{4}tuu_{tx} - \frac{1}{4}uu_x - \frac{1}{4}xuu_{2x} \\
\mathrm{T}_2 &= \frac{1}{2}xuu_x^2 + \frac{1}{2}xuu_{xt} + \frac{1}{3}xu^2u_{2x} + \frac{1}{4}tu_t^2 + \frac{1}{4}xu_tu_x - \frac{1}{4}uu_t - \frac{1}{4}tuu_{2t} - \frac{1}{3}tu^2u_{tx} - \frac{1}{3}u^2u_x
\end{aligned}$$

case3: $\xi_3 = tu_t + u$

$$\begin{aligned}
\mathrm{T}_1 &= \frac{1}{4}tuu_{xt} + \frac{1}{6}tuu_x^2 + \frac{1}{3}tu^2u_{2x} + \frac{1}{4}tu_tu_x + \frac{1}{4}uu_x - \frac{1}{4}uu_x, \\
\mathrm{T}_2 &= \frac{1}{3}tuu_xu_t + \frac{2}{3}u^2u_x - \frac{1}{2}uu_t - \frac{1}{4}tuu_{2t} + \frac{1}{4}tu_t^2 + \frac{1}{4}uu_t - \frac{1}{3}tu^2u_{tx} - \frac{2}{3}u^2u_x
\end{aligned}$$

case4: $\xi_4 = u_t$

$$\begin{aligned}
\mathrm{T}_1 &= \frac{1}{2}uu_{xt} + \frac{1}{6}uu_x^2 + \frac{1}{3}u^2u_{2x} - \frac{1}{4}uu_{tx} + \frac{1}{4}u_xu_t, \\
\mathrm{T}_2 &= -\frac{1}{4}uu_t + \frac{1}{4}u_t^2 - \frac{1}{3}u^2u_{tx} + \frac{1}{3}uu_xu_t
\end{aligned}$$

The Lie point symmetry generators of equation (1) are given by

(14)
$$\begin{aligned}
&\mathbf{X}_1 = \partial_x,\ \mathbf{X}_2 = \partial_t,\ \mathbf{X}_3 = x\partial_x + u\partial_u, \\
&\mathbf{X}_4 = t\partial_t - u\partial_u,\ \mathbf{X}_5 = tx\partial_x + \frac{1}{2}t^2\partial_t + x\partial_u
\end{aligned}$$

We now observe a symmetry analysis of the multipliers under the generators $X_1, \cdots, X_5$ as follows in table 1. So by proposition(III) the action of the generator $\mathbf{X}_1, \mathbf{X}_3, \mathbf{X}_4$, and $\mathbf{X}_5$ results in new multipliers

$$\begin{aligned}
&Q_1 = t\xi_4 - 1,\ Q_2 = u_x,\ Q_3 = xtu_x,\ Q_4 = xu_x \\
&Q_5 = u,\ Q_6 = t^2\xi_4,\ Q_7 = t\xi_4,\ Q_8 = t\xi_4 - u, \\
&Q_9 = \frac{1}{2}t^3\xi_4 + t^2xu_x - tx,\ Q_{10} = txu_x + \frac{1}{2}t^2\xi_4 \\
&Q_{11} = \frac{1}{2}t^2\xi_4 + x.
\end{aligned}$$

and thus would result in new conservation laws for equation (1).

Table I. algebra Lie of table Commutation

$\mathbf{X}_i, \xi_j$	ξ_1	ξ_2	ξ_3	ξ_4
$\mathbf{X}_1$	$t\xi_4 - 1$	u_x	0	0
$\mathbf{X}_2$	ξ_2	ξ_4	ξ_4	0
$\mathbf{X}_3$	xtu_x	xu_x	u	0
$\mathbf{X}_4$	$t^2\xi_4$	$t\xi_4$	$t\xi_4 - u$	0
$\mathbf{X}_5$	$\frac{1}{2}t^3\xi_4 + t^2xu_x - tx$	$txu_x + \frac{1}{2}t^2\xi_4$	$\frac{1}{2}t^2\xi_4 + x$	0

Conclusions

We utilized Euler operator to construct determining equation for multipliers.Then, higher order conservation laws of the HSE constructed by applying the 2-dimensional Homotopy formula. Furthermore, by analyzing the symmetry action of the Lie point symmetry generators on the multipliers for HSE Obtained new conservation laws from known conservation laws.

References

[1] S.C. ANCO, G.W. BLUMAN, Direct construction method for conser- vation laws of partial differential equations. Part II: General treatment, Eur. J. Appl. Math. 13 (2002) , 567585.
[2] G.W. BLUMAN, New conservation laws obtained directly from symmetry action on a known conservation law, J. Math. Anal. Appl. 322 (2006), 233-250.
[3] G.W. BLUMAN, A.F. CHEVIACOV, S.C. ANCO, Applications of Symmetry Methods to Partial Differential Equations, Applied Mathematical Sciences, vol. 168 Springer, New York, 2010.
[4] J.K. Hunter and R.A. Saxton, SIAM J. Appl. Math. 51 (1991) 1498.
[5] M. NADJAFIKHAH, V. SHIRVANI-SH, Lie symmetries and conservation laws of the Hirota-Ramani equation, Commun. Nonlinear Sci. Numer. Simul. 17 (2012), 40644073.
[6] P.J. OLVER, Applications of Lie Groups to Differential Equations, In: Graduate Texts in Mathematics, vol. 107. Springer, New York, 1993.
[7] D. Poole, W. HEREMAN, Symbolic computation of conservation laws of nonlinear partial differential equations using homotopy operators, Ph.D. dissertation, Colorado School of Mines, Golden, Colorado, (2009).
[8] D. Poole, W. HEREMAN, The homotopy operator method for symbolic integration by parts and inversion of divergences with applications, Appl. Anal. 87 (2010), 433-455.
[9] E.G. Virga, Variational Theories for Liquid Crystals, Chapman Hall, London (1994).
[10] [8] J.K. Hunter and R.A. Saxton, SIAM J. Appl. Math. 51 (1991) 1498.