New nonstandard finite difference scheme for COVID–19 model

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***Abstract*— The target of this paper is to implement numerical solution of the Coronavirus disease 2019 (COVID–19) model based on nonstandard finite difference (NSFD) scheme. First, the positivity and boundedness of the model are discussed. In continuation, the stability analysis of the equilibrium points the proposed model are discussed. Afterwards, the NSFD scheme is implemented to study the dynamic behaviours COVID–19 model. Numerical results show the efficiency of the proposed NSFD when is applied to COVID–19 model.**

Keywords*—*COVID–19 model, nonstandard finite difference scheme, positivity, boundedness, stability.

I. Introduction

Mathematical modelling is benefit decision tool which can be applied to control human diseases. Coronavirus disease is an infections disease appearing in China, in the city of Wuhan on Dec 31, 2019. The disease has rapidly spread spread in other countries. At the onset of COVID–19, patients usually show symptoms connected to viral pneumonia, usually fever, cough, sore throat myalgia and fatigue. The virus can spread from an individual to another through respiratory droplets and close contact. In mathematics modelling provides a tool to better understand the transmission dynamics of infections disease [**1**, **2**]. In this framework, the changes in populations of several classes of interacting individuals can be described by a system of ordinary differential equations (ODEs). Finding the exact solutions of nonlinear system ODEs in many cases is difficult. Hence, it is necessary to find their efficient numerical solutions by using some numerical methods. Whenever a continuous system of ODEs has been converted into difference system, the properties of the continuous system is not transferred completely to the difference system in the case of choosing large stepsize in the difference system. However, if we use NSFD scheme it leads to the properties of the continuous system can be preserved into its difference system. The NSFD schemes are extended for compensating the weakness, such as numerical instability that may be caused by standard finite difference schemes. In the present paper, we apply a susceptible, infections, quarantined and recovered (SIQR) framework to model the dynamic of COVID–19. A reasonable model for the COVID–19 at time t may be described by the following equations

$$\begin{cases} \dfrac{\mathrm{d}S}{\mathrm{d}t} = -\beta\dfrac{SI}{N} - k_0 S, \\ \dfrac{\mathrm{d}I}{\mathrm{d}t} = \beta\dfrac{SI}{N} - (\alpha+\eta)I, \\ \dfrac{\mathrm{d}Q}{\mathrm{d}t} = \eta I - \gamma Q, \\ \dfrac{\mathrm{d}R}{\mathrm{d}t} = \gamma Q + \alpha I, \\ S(0) = S_0,\ I(0) = I_0,\ Q(0) = Q_0,\ R(0) = R_0. \end{cases} \tag{1}$$

In this model, $S(t)$ is the number of susceptible population in the community of time t, $I(t)$ denotes the number of infected people at time t, $Q(t)$ stands the number of quarantined population in the community of time t and $R(t)$ presents the number of recovered individuals at time t. Here the positive parameters β, k_0, α, η and γ denote the infection rate, the containment rate, the recovering of asymptomatic individuals, detection of infected individuals and recovering of quarantined individuals, respectively. Since, the exact solution of the system (1) cannot be easily obtained, therefore a numerical approach is used.

This paper is structured as follows. In Section 2, we prove positivity and boundedness of the solution model (1). Section 3, devotes to stability analysis of the equilibrium points of proposed COVID–19 model. In Section 4, we present a summary of important feature for constructing NSFD schemes for system of ODEs. In the end, numerical results are given in Section 5.

II. Positivity and boundedness of the COVID–19 model

In this part, we are going to prove positivity and boundedness of the solution model of (1). First observe that, the fourth equation of system (1) is not coupled with the other equations. As a result, we can consider the first three equations of the system (1).

Theorem 1. *If $S(0),\ I(0)$ and $Q(0) > 0$, then for all $t \geq 0$, $S(t),\ I(t)$ and $Q(t) > 0$.*

Proof. Since the IQ and SQ coordinate planes are invariant under the flows of system, hence, for all $t \geq 0$, $S(t)$ and $I(t) > 0$. Let $A = \{t \geq 0 : \ Q(t) < 0\}$, we will show that $A = \emptyset$. Suppose that $A \neq \emptyset$ and $t_1 = \min A$. Since $Q(t)$ is continuous and $Q(0) > 0$, so $t_1 > 0$. Hence for all $t \in [0, t_1)$, we have $Q(t) > 0$. Moreover, from $Q(t_1) = 0$ and the third equation of system (1) it follows that $Q'(t_1) = \eta I(t_1) - \gamma Q(t_1) > 0$. Hence, there exists $\epsilon > 0$ such that $Q'(t) > 0$, for all $t \in (t_1 - \epsilon, t_1 + \epsilon) \subseteq (0, \infty)$. It implies $Q(t_1) > Q(t_1 - \frac{\epsilon}{2}) > 0$ which contradicts $Q(t_1) = 0$. □

Theorem 2. *If $S(0),\ I(0)$ and $Q(0) > 0$, then for all $t \geq 0$, $S(t) + I(t) + Q(t) \leq S(0) + I(0) + Q(0)$.*

Proof. Setting $K(t) = S(t) + I(t) + Q(t)$, hence for all $t \geq 0$, $K'(t) = -k_0 S(t) - \alpha I(t) - \gamma Q(t) \leq 0$. Therefore for all $t \geq 0$, $K(t) \leq K(0)$ which implies $S(t) + I(t) + Q(t) \leq S(0) + I(0) + Q(0)$. □

III. Stability analysis of the COVID–19 model

In order to evaluate the equilibrium points of the system Eq. (1), let

$$\begin{cases} -\beta\dfrac{SI}{N} - k_0 S = 0, \\ \beta\dfrac{SI}{N} - (\alpha+\eta)I = 0, \\ \eta I - \gamma Q = 0. \end{cases}$$

Hence the equilibrium points are $E_1 = (0,0,0)$ and $E_2 = (S^*, I^*, Q^*)$, where

$$S^* = \frac{(\alpha+\eta)N}{\beta}, \qquad I^* = -k_0\frac{N}{\beta}, \qquad Q^* = -\frac{\eta k_0}{\gamma\beta}N.$$

Theorem 3. *(i) System (1) is always locally asymptotically stable around E_1.*
(ii) System (1) is always unstable around E_2.

Proof. At the equilibrium point E_1, the Jacobian matrix is given by

$$J(E_1) = \begin{bmatrix} -k_0 & 0 & 0 \\ 0 & -(\alpha+\eta) & 0 \\ 0 & 0 & -\gamma \end{bmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = -k_0$, $\lambda_2 = -(\alpha+\eta)$ and $\lambda_3 = -\gamma$. Since $\lambda_i < 0$, $i = 1,2,3$, therefore the equilibrium point E_1 is asymptotically stable. At the equilibrium point E_2 , the Jacobian can be written as

$$J(E_2) = \begin{bmatrix} 0 & -\dfrac{\beta S^*}{N} & 0 \\ \dfrac{\beta I^*}{N} & 0 & 0 \\ 0 & \eta & -\gamma \end{bmatrix}.$$

Hence, the characteristic polynomial $P(\lambda)$ is given by $P(\lambda) = (\lambda+\gamma)(\lambda^2 - k_0(\alpha+\eta))$. Since $k_0(\alpha+\eta) > 0$, therefore one of the roots $P(\lambda)$ is positive and consequently the equilibrium point E_2 is always unstable. □

IV. A NSFD scheme for the COVID–19 model

The NSFD schemes were firstly proposed by Mickens. Consider an initial value problem in the following form

$$X'(t) = f(X(t)), \qquad X(t_0) = X_0,\ t \in [t_0, t_f]. \tag{2}$$

A NSFD scheme is constructed by the following two steps. The first, the derivative in the Eq. (2) is replaced by a discrete form $X'(t_k) \simeq \frac{X_{k+1}-X_k}{\phi(h)}$, where X_k is an approximation of $X(t_k)$ and $0 < \phi(h) < 1$ with $\phi(h) = h + O(h^2)$. The second structure of the NSFD scheme requires that the dependent functions must be modeled on the discrete time computational grid [3]. For example, the terms xy and x^2 can be approximated using $x_n y_{n+1}$ and $x_n x_{n+1}$, respectively. Applying the NSFD scheme, we obtain the following discrete model for the system Eq. (1)

$$\begin{cases} \dfrac{S_{k+1}-S_k}{\phi_1(h)} = -\dfrac{\beta I_k S_{k+1}}{N} - k_0 S_{k+1}, \\ \dfrac{I_{k+1}-I_k}{\phi_2(h)} = \dfrac{\beta S_{k+1} I_k}{N} - (\alpha+\eta) I_{k+1}, \\ \dfrac{Q_{k+1}-Q_k}{\phi_3(h)} = \eta I_{k+1} - \gamma Q_k. \end{cases}$$

With doing some computations, we get

$$\begin{cases} S_{k+1} = \dfrac{S_{k+1}}{1+\beta\phi_1 \frac{I_k}{N} + k_0\phi_1}, \\ I_{k_0+1} = \dfrac{I_k + \beta\phi_2 \frac{S_{k+1} I_k}{N}}{1+(\alpha+\eta)\phi_2}, \\ Q_{k+1} = \dfrac{Q_k + \phi_3 \eta I_{k+1}}{1+\gamma\phi_3}, \end{cases} \tag{3}$$

where

$$\phi_1(h) = \frac{e^{k_0 h}-1}{k_0}, \quad \phi_2(h) = \frac{e^{(\alpha+\eta)h}-1}{\alpha+\eta}, \quad \phi_3(h) = \frac{e^{\gamma h}-1}{\gamma}.$$

The Eq. (3) must be computed in sequence, because the value of S_{k+1} is used for calculating the value I_{k+1}, which is then used to calculate the value of Q_{k+1}. Note that the right hand side of (3) is always positive for all stepsize h. This shows that the numerical solution of (3) is always positive with any positive initial value and stepsize.

V. Numerical results

In the present section, the numerical solutions of the proposed NSFD scheme (3) on two cases are investigated. First, we consider the parameter values $\beta = 0.32$, $\alpha = \eta = 0.018$, $k_0 = 0.033$ and $\gamma = 0.02$ with initial condition $S_0 = 20$, $I_0 = 30$, $Q_0 = 25$ for simulating time $1000s$ and stepsize $h = 2$. Figure 1 shows that the NSFD scheme (3) converges to the equilibrium point $E = (0, 0, 0)$. In Figure 2 the numerical solutions of the NSFD scheme are plotted by choosing $k_0 = 0.022$, $\beta = 0.3194$, $\eta = \alpha = 0.0206$ and $\gamma = 0.02$ with the initial condition $S_0 = 20$, $I_0 = 30$, $Q_0 = 25$ and stepsize $h = 0.5$. The Figure 2 confirms that (S_k, I_k, Q_k) converges to the equilibrium point $E = (0, 0, 0)$.

VI. Conclusion and future work

In this paper, the positivity and boundedness properties of a COVID–19 model was analyzed. Afterwards, the stability analysis of the equilibrium points of the COVID–19 model was investigated. In continuation, a numerical method based on the NSFD scheme was applied to the solution of the COVID–19 model. To ascertain, the accuracy and efficacy of the proposed NSFD scheme, some numerical results were presented. Future work focus on constructing superior NSFD scheme for numerical solution of the fractional–order COVID–19 model.

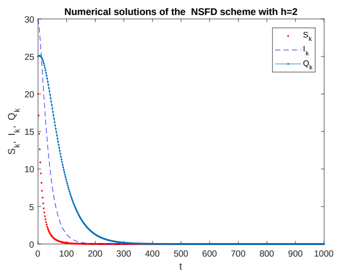

Fig. 1. Numerical simulation with $h = 2$ for the NSFD scheme (3).

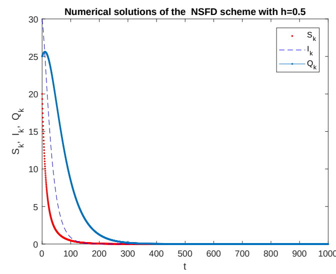

Fig. 2. Numerical simulation with $h = 0.5$ for the NSFD scheme (3).

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