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Travelling wave solution of Balitsky-Kovchegov equation

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Abstract. In this work, we suggest an approximate analytical solution of the Balitsky-Kovchegov (BK) equation in momentum space using a method called the homotopy perturbation method (HPM). With some change of variables and the truncation of the BFKL (Balitsky-Fadin-Kuraev-Lipatov) kernel, the BK equation in momentum space can be transformed into the FKPP (Fisher-Kolmogorov-Petrovsky-Piscounov) equation. Observed geometric scaling at small-x at HERA and travelling wave solution of the FKPP equation are similar. The solution of the BK equation obtained in this work also suggests the travelling wave nature of the measured scattering amplitude N(k, Y) plotted at various rapidities. The solution obtained in his work can be helpful in further phenomenological studies at high-density QCD.

Keywords: Parton saturation, BK equation, Travelling wave solution

1 Introduction

One of the most important phenomena of QCD at high energies or equivalently at small-x (Bjorken x) is the growth of hadronic cross-sections. Hadronic cross-sections have been incited by states with high partonic densities at small-x. Many theoretical and phenomenological efforts have been made to understand and explain the high-density QCD at small-x. Let us talk only about gluon densities at very small-x as one can neglect quark densities there. The fast growth of gluons at small-x is well described by the BFKL equation [1, 2]. This equation can be derived with perturbative QCD (pQCD) by resuming leading logarithms of energies expressed in terms of x such as $\ln(1/x) \gg \ln(Q^2/\mu^2)$, where Q and μ being the photon virtuality and renormalization scale respectively. It is seen from the solution of the BFKL equation that the measured scattering amplitude N(k, Y) (k being the transverse momentum and Y being the rapidity of evolved gluons) and hence the total cross-section exhibits an exponential growth with rapidity Y. At very small-x, the rapidly increased gluon densities need to be tamed down to hold the unitarity and hence Froissart-Martin bound [3]. Froissart-Martin bound says, the total cross-section of a given process cannot grow faster than the logarithm of energy squared. Thus, at very small x or high energies, the BFKL equation violates the unitarity and hence Froissart-Martin bound. So, its applicability is limited and cannot be used at arbitrarily high energies.

The above problems faced by the BFKL equation are to be addressed to understand the physics at small-x. The solution is that at high energies or small-x, gluons themselves start to recombine and get saturated finally. The first idea of gluon-gluon recombination is addressed in ref. [4-8]. The gluon-gluon recombination will tame down the gluon density and saturation of gluons will solve the unitarity problem. BFKL equation, being linear, could not address the nonlinear effect of gluon recombination and saturation and hence is unable to explain implicit physics at high-density QCD.

It is imperative to understand the implicit physics in the saturation region of gluons at small-x. In this region, linear QCD evolution equations are replaced by the nonlinear QCD evolution equations, helping to understand the gluon-gluon recombination and saturation effect. The nonlinear evolution equations have important features dealing with the saturation effect as they contain damping terms that reflect the saturation effect arising out of gluon-gluon recombination. The Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK) equation [9-12] permits gluon saturation in high-density gluon region at small-x that addresses the nonlinear correction using the Wilson renormalization group approach. However, it is complicated to solve the JIMWLK equation because of its complex nature, and hence unable to apply it in phenomenological studies. Instead, its mean filed approximation BK equation [13-16] is studied most in the context of saturation effect. Though, it is tough to solve the BK equation using general methods. BK equation is an integrodifferential equation in coordinate space that can be transformed into momentum space resulting in a partial differential equation. Analytical solutions to the BK equation proposed recently with different approaches using some approximations can be found in ref. [17-22]. These analytical solutions shed light on the ability of the BK equation in explaining gluon saturation and its application in the high-energy hadron scattering phenomena.

In this work, we suggest an approximate analytical solution to the BK equation using the homotopy perturbation method (HPM) [23, 24]. The BK equation in momentum space with some change of variables and truncation of the BFKL kernel can be transformed into the Fisher-Kolmogorov-Petrovsky-Piscounov (FKPP) equation [17-19]. The FKPP equation [25, 26] is a partial differential equation that belongs to the reaction-diffusion equation in statistical physics. Observed geometric scaling phenomena at small-x at HERA can be related to the travelling wave solution of the FKPP equation [27]. The transition of the scattering amplitude into the saturation region is similar to the formation of the front of the travelling wave of the FKPP equation [17]. We obtain the solution of the BK equation which also suggests the travelling wave nature of the solution. The solution of the BK equation obtained in this work can be helpful for further phenomenological studies in light of present and future accelerator facilities.

We organize the paper as follows. In section 2, we discuss the relation between the BK and the FKPP equations. The solution of the BK equation is presented in section 3. The summary and conclusion are presented in section 4.

2 Relation between the BK and FKPP equations

The relation between the BK and FKPP equations has been found in the pioneering work done by S. Munier and R. B. Peschanski [17-19]. In this section, we discuss how to relate the BK equation with the FKPP equation following their work.

BK equation is about energy dependence of scattering amplitude at small-x, it is often convenient to carry out work in the pQCD dipole picture of deep inelastic scattering (DIS) [28-31]. This picture is known as the dipole model, which is valid at small-x. The main advantage of the dipole picture of DIS is the factorization of the scattering process into several steps, resulting in smooth use of pQCD. In the dipole picture, an incoming virtual photon after fluctuation changes to a quark-antiquark dipole. The quark-antiquark pair then scattered off the target proton and recombines to form some final state particles. In reference to the dipole picture of DIS, in the leading logarithm approximation of pQCD, the cross-section in terms of the total rapidity (Y) and the virtuality of the photon (Q) factorizes to [27]

$$\sigma^{\gamma*p} = \int_0^\infty x_{01} dx_{01} \int_0^1 dz |\psi(z, x_{01}Q)|^2 N(Y, x_{01}) \,, \tag{1}$$

where z being the longitudinal momentum fraction of the quark of the virtual photon, $\psi(z, x_{01}Q)$ is the photon wave function on a quark-antiquark dipole of its size x_{01}. $N(Y, x_{01})$ is the forward dipole-proton scattering amplitude.

Within the large N_c approximation at fixed coupling and for a homogeneous nuclear target, the measured scattering amplitude $N(k, Y)$ at transverse momentum k and total rapidity Y obeys the BK equation in momentum space given by [15]

$$\partial_Y N = \bar{\boldsymbol{\alpha}}\, \chi(-\partial_L) N - \bar{\boldsymbol{\alpha}}\, N^2 \,, \tag{2}$$

where $\bar{\boldsymbol{\alpha}} = \frac{\alpha_s N_c}{\pi}$ and $\chi(\zeta) = 2\psi(1) - \psi(\zeta) - \psi(1-\zeta)$ is the BFKL kernel. $\zeta = -\partial_L$, where $L = \ln\left(\frac{k^2}{k_0^2}\right)$, k_0 being some low momentum scale at fixed. The expansion of the BFKL kernel around $\zeta = \frac{1}{2}$ has been suggested in ref. [17], and with this expansion equation (2) reduces to the nonlinear partial differential equation given by

$$\partial_Y N = \bar{\boldsymbol{\alpha}}\, \bar{\chi}(-\partial_L) N - \bar{\boldsymbol{\alpha}}\, N^2 \,, \tag{3}$$

where

$$\bar{\chi}(-\partial_L) = \chi\left(\frac{1}{2}\right) + \frac{\chi''\left(\frac{1}{2}\right)}{2}\left(\partial_L + \frac{1}{2}\right)^2. \tag{4}$$

In reference to the above expansion and defining $\bar{\zeta} = 1 - \frac{1}{2}\sqrt{1 + 8\frac{\chi\left(\frac{1}{2}\right)}{\chi''\left(\frac{1}{2}\right)}}$, with the following change of variables [17]

$$t = \frac{\bar{\boldsymbol{\alpha}}\,\chi''\left(\frac{1}{2}\right)}{2}(1-\bar{\zeta})^2 Y, \quad x = (1-\bar{\zeta})\left(L + \frac{\bar{\boldsymbol{\alpha}}\,\chi''\left(\frac{1}{2}\right)}{2} Y\right),$$

$$u(t,x) = \frac{2}{\chi''\left(\frac{1}{2}\right)(1-\bar{\zeta})^2} N\left(\frac{2t}{\bar{\boldsymbol{\alpha}}\,\chi''\left(\frac{1}{2}\right)(1-\bar{\zeta})^2}, \frac{x}{(1-\bar{\zeta})} - \frac{t}{(1-\bar{\zeta})^2}\right),$$

the equation (3) turns into the FKPP equation for $u(t,x)$, can be expressed as [17]

$$\partial_t u(t,x) = \partial_x^2 u(t,x) + u(t,x) - u^2(t,x). \tag{5}$$

Thus, with some change of variable transformation, it is seen that the BK equation (3) can be transformed to the above equation (5), which is the famous FKPP equation.

3 Solution of the BK equation with HPM

Given the discussion of the relation between the FKPP and the BK equations discussed in the previous section, let us solve the BK equation for the scattering amplitude $N(k,Y)$. The solution of the BK equation (3) in connection with the equation (5) for the scattering amplitude $N(k,Y)$ using the HPM can be written as

$$N(k,Y) = \frac{N_0 e^Y}{1 - N_0 + N_0 e^Y} \ , \tag{6}$$

where N_0 is the initial condition. Once the initial condition is known to us, the solution of the BK equation gives the scattering amplitude $N(k,Y)$ at any given rapidity $Y > 0$. In this work, we will use the following initial condition given by K. Golec-Biernat and M. Wusthoff (GBW), introduced first in ref. [32]

$$N^{GBW}(r, Y=0) = 1 - exp\left[-\left(\frac{r^2 Q_{s0}^2}{4}\right)\right]. \tag{7}$$

Q_{s0}^2 is the fit parameter, called the initial saturation scale squared. This initial condition can simply be Fourier transformed into momentum space analytically and applied to the BK equation in momentum space. The momentum space result of the GBW initial condition can be written as

$$N^{GBW}(k, Y=0) = \int \frac{d^2 r}{2\pi r^2} e^{ik.r} N^{GBW}(r, Y=0)$$

$$= \frac{1}{2}\Gamma\left(0, \frac{k^2}{Q_{s0}^2}\right). \tag{8}$$

$\Gamma(0, k^2/Q_{s0}^2)$ is the incomplete gamma function. At large values of k^2/Q_{s0}^2, this behaves as

$$\Gamma\left(0, \frac{k^2}{Q_{s0}^2}\right) = exp\left(-\frac{k^2}{Q_{s0}^2}\right).$$

Therefore,

$$N^{GBW}(k, Y=0) = \frac{1}{2}\ exp\left(-\frac{k^2}{Q_{s0}^2}\right) . \tag{9}$$

Substitution of the above equation in equation (6) for the initial condition N_0, we obtain the scattering amplitude $N(k,Y)$ with GBW as the initial condition

$$N(k,Y) = \frac{e^{Y - k^2/Q_{s0}^2}}{1 - e^{-k^2/Q_{s0}^2} + e^{Y - k^2/Q_{s0}^2}} \ . \tag{10}$$

This is the approximate analytical solution of the BK equation (3). The evolution of the scattering amplitude at different rapidities can be seen in Fig. 1

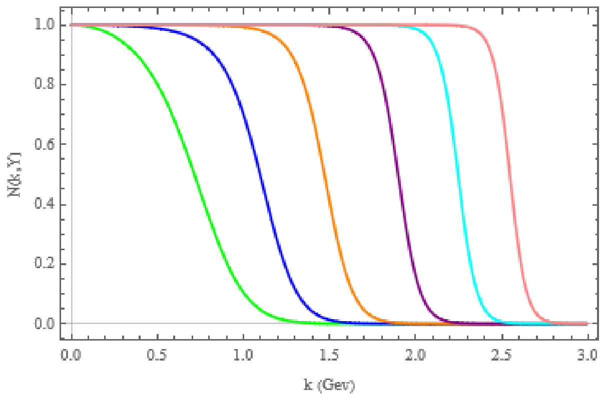

Figure 1. The solution of the BK equation in momentum space, $N(k)$, at various rapidities Y = 2, Y = 5, Y = 9, Y = 15, Y = 21 and Y = 27.

4 Summary and conclusion

This work has suggested an approximate analytical solution of the BK equation using the HPM. The relation between the geometric scaling phenomena of the solution of the BK equation and the travelling wave solution of the FKPP equation, as suggested by S. Munier and R. Peschanski in their pioneering work, has guided the scientific community working in the field of gluon saturation. In this work, we have started our discussion with the relation between the BK and FKPP equations. We carried out work in the pQCD dipole picture of DIS in which the measured scattering amplitude $N(k, Y)$ obeys the BK equation in the momentum space frame to work in the context of at least travelling wave solution and the geometric scaling. Afterward, with some change of variables and a slight approximation in the BK equation, we ended with the approximated analytical solution of the BK equation in the momentum space. We plotted the obtained solution, equation (10), at different rapidities in Fig. 1 to check the travelling wave nature of the solution. Indeed, one can see the solution's travelling wave nature. It indicates that at high energies, the scattering amplitude behaves as a wave travelling from the region $N = 1$ to $N = 0$ as k increases without being changed in the profile. It is indeed a vital physical result of this travelling wave approach.

The solution obtained in this work can be helpful in further phenomenological studies in high-density QCD and saturation regions. However, it is going to be interesting to observe whether this type of travelling wave solution and geometric scaling exist or not at very high energies when EIC (Electron-Ion-Collider) [33] and other future projects run operations. Nevertheless, the BK equation with truncation of the BFKL kernel successfully explains the observed geometric scaling and the travelling wave nature of its solution at current accelerator facilities. We must rely on the future acceleration facilities for precise measurements of observed phenomena and their confirmation.

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